QCD calculations for boosted top production at hadron colliders

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Factorization and resummation for boosted top production

Collaboration with Andrea Ferroglia, Li Lin Yang, and Simone Marzani

1) Factorization and resummation for large pair invariant mass distribution (arXiv:1205.3662)

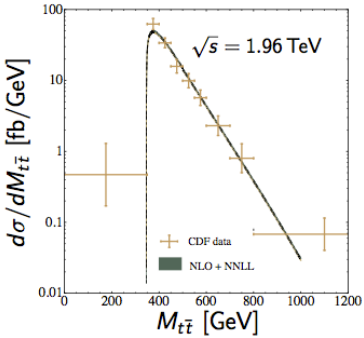
2) Four Wilson-line soft function to NNLO (arXiv:1207.4798)

3) An NNLO soft plus virtual approximation for $d\sigma/dM_{t\bar{t}}$ in $m_t \ll M_{t\bar{t}}$ limit (arXiv:1306.1537)

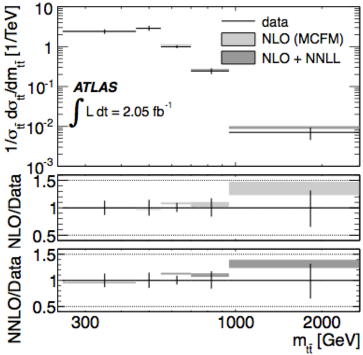
4) Factorization and resummation for single-particle inclusive distributions (i.e. $d\sigma/dp_T$) (arXiv:1310.3836)

Production at high top-pair invariant mass

Tevatron $\sqrt{s} \approx 2\ TeV$



LHC: $\sqrt{s} = 7\ TeV$



- LHC has data in "boosted regime" $M_{t\bar{t}} \gg m_t$
- not just "corner of phase space": important for new physics searches

QCD corrections in boosted top production

Consider very large pair invariant mass where $\tau = M_{t\bar{t}}^2/s \to 1$

$$\frac{d\sigma}{dM_{t\bar{t}}}(s, m_t, M_{t\bar{t}}) = \sum_{i,j} \int_\tau^1 \frac{dz}{z}\, ff_{ij}(\tau/z, \mu_f)\, \frac{d\hat{\sigma}_{ij}}{dM_{t\bar{t}}}(z, m_t, M_{t\bar{t}}, \mu_f)$$

$$ff_{ij}(y, \mu_f) = \int_y^1 \frac{dx}{x}\, f_{i/h_1}(x, \mu_f)\, f_{j/h_2}(y/x, \mu_f)$$

Two kinds of large logarithms appear:

- soft logs: $[\ln(1-z)/(1-z)]_+ \qquad (z \equiv M_{t\bar{t}}^2/\hat{s})$
- small-mass (collinear) logs: $\ln m_t/M_{t\bar{t}}$

Goal: set up a framework which can factorize cross sections and resum both types of logs, i.e. understand factorization in double soft and small-mass limit

Reference point: factorization of partonic cross sections in the soft limit

1) Pair invariant mass distributions (PIM kinematics i.e. $d\sigma/dM$)

$$d\hat{\sigma}_{ij}(z, M, m_t, \cos\theta, \mu_f) = \mathrm{Tr}\left[\mathbf{H}^m_{ij}(M, m_t, t_1, \mu_f)\,\mathbf{S}^{m,\mathrm{PIM}}_{ij}(\sqrt{\hat{s}}(1-z), m_t, t_1, \mu_f)\right] + \mathcal{O}(1-z)$$

2) Single particle inclusive distributions (1PI kinematics i.e. $d\sigma/dp_T$)

$$d\hat{\sigma}_{ij}(s_4, \hat{s}, \hat{t}_1, \hat{u}_1, m_t, \mu_f) = \mathrm{Tr}\left[\mathbf{H}^m_{ij}(\hat{s}, \hat{t}_1, \hat{u}_1, m_t, \mu_f)\,\mathbf{S}^{m,\mathrm{1PI}}_{ij}(s_4, \hat{s}, \hat{t}_1, \hat{u}_1, m_t, \mu_f)\right] + \; O\left(\frac{s_4}{m_t^2}\right)$$

- both cases known to NNLL [Ahrens, Ferroglia, Neubert, BP, Yang; Kidonakis], enough for all soft logs and μ-dependent terms at NNLO
- In small-mass limit both $\mathbf{H}^m$ and $\mathbf{S}^m$ have logarithms in $\ln m_t^2/\hat{s}$

PIM kinematics: factorization with fragmentation functions in small-mass limit

Idea: start with well-known factorization theorem for energetic top-quark production [Mele and Nason 1990]

Generic heavy-quark production cross section (i.e. $e^+e^- \to t + X$)

$$\frac{d\sigma_t}{dz}(z, m_t, \mu) = \sum_a \int_z^1 \frac{dx}{x} \frac{d\tilde{\sigma}_a}{dx}(x, m_t = 0, \mu) D_{a/t}^{(n_l+n_h)}\left(\frac{z}{x}, m_t, \mu\right)$$

- a lot like factorization with PDFs, just apply to our case
- then weave together with soft limit of each function [arXiv:1205.3662]

caveat: closed heavy-quark loops proportional to powers of N_h complicate things, leave aside in rest of talk

First layer: the small-mass limit

1) When $m_t \ll M \equiv M_{t\bar{t}}$ $\qquad$ $\left(f(z) \otimes g(z) \equiv \int_z^1 \frac{dx}{x} f(x)\, g(z/x)\right)$

$$\frac{d\hat{\sigma}}{dM}(M, m_t, z, \mu) \sim \frac{d\hat{\sigma}}{dM}(M, z, m_t = 0, \mu) \otimes D^2_{t/t}(m_t, z, \mu) + \mathcal{O}\left(\frac{m_t^2}{M^2}\right)$$

- m_t dependence factorized into heavy-quark fragmentation function $D_{t/t}$ (collinear emissions)
- M dependence factorized into massless partonic cross section (wide-angle emissions)

- next step: factorize $d\hat{\sigma}/dM$ and $D_{t/t}$ in soft limit

Second layer: soft limit ($z \to 1$)

1) Massless partonic cross section

$$d\hat{\sigma}_{ij}(z, M, t_1, \mu_f) = \mathrm{Tr}\left[\mathbf{H}_{ij}(M, t_1, \mu_f)\,\mathbf{S}_{ij}(\sqrt{\hat{s}}(1-z), t_1, \mu_f)\right] + \mathcal{O}(1-z)$$

- $\mathbf{H}_{ij}$ related to virtual corrections to massless 2→2 scattering
- $\mathbf{S}_{ij}$ related to soft real corrections to massless 2→2 scattering

2) Fragmentation function

$$D_{t/t}^{(n_l)}(z, m_t, \mu_f) = C_D(m_t, \mu_f)\, S_D(m_t(1-z), \mu_f) + \mathcal{O}(1-z)$$

- C_D related to virtual corrections to fragmentation function
- S_D related to real corrections to fragmentation function (and proposed to be equivalent to perturbative shape function [Gardi '05 ;Neubert '07])

Factorization at NNLO

The factorized cross section is

$$\frac{d\hat{\sigma}}{dM} \sim \mathrm{Tr}[\mathbf{H}(M,\mu)\mathbf{S}(M(1-z),\mu)] \otimes C_D^2(m_t,\mu) S_D^2(m_t(1-z),\mu) + \mathcal{O}(1-z) + \mathcal{O}\left(\frac{m_t^2}{M^2}\right)$$

All component functions known at NNLO

- C_D and S_D from fragmentation function for generic z [Melnikov, Mitov '04]
- **H** from virtual corrections to $gg, q\bar{q} \to \bar{Q}Q$ scattering [Glover et. al '00-'01] after IR renormalization procedure [Ferroglia, BP, Yang '13]
- **S from real emission corrections to** $gg, q\bar{q} \to \bar{q}' q'$ **in soft limit** [Ferroglia, BP, Yang '12]

If NNLO corrections are large because of logarithmic corrections, can use renormalization group to resum them to NNLL accuracy (standard SCET methods, momentum space or otherwise)

A step back

The factorized cross section is

$$\frac{d\hat{\sigma}}{dM} \sim \mathrm{Tr}[\mathbf{H}(M,\mu)\mathbf{S}(M(1-z),\mu)] \otimes C_D^2(m_t,\mu)S_D^2(m_t(1-z),\mu) + \mathcal{O}(1-z) + \mathcal{O}\left(\frac{m_t^2}{M^2}\right)$$

- we derived this by taking $m_t \to 0$ limit then soft limit
- natural question: how to do it the other way around?
- in other words, would like to start with

$$d\hat{\sigma}_{ij}(z,M,m_t,\cos\theta,\mu_f) = \mathrm{Tr}\left[\mathbf{H}_{ij}^m(M,m_t,t_1,\mu_f)\,\mathbf{S}_{ij}^{m,\mathrm{PIM}}(\sqrt{\hat{s}}(1-z),m_t,t_1,\mu_f)\right] + \mathcal{O}(1-z)$$

  and subfactorize $\mathbf{H}_{ij}^m$ and $\mathbf{S}_{ij}^m$ in $m_t^2/\hat{s}$ limit...
- simple for hard function (next), harder for soft function (later)

Factorized hard function in $m_t^2/\hat{s} \ll 1$ limit

Three pieces

$$|\mathcal{M}(\epsilon, M, m_t, t_1)\rangle = Z_{[q]}(\epsilon, m_t, \mu)\,|\mathcal{M}(\epsilon, M, t_1)\rangle \quad \text{[Mitov/Moch]}$$

$$\lim_{\epsilon\to 0} \mathbf{Z}_m^{-1}(\epsilon, M, m_t, t_1, \mu)\,|\mathcal{M}(\epsilon, M, m_t, t_1)\rangle = |\mathcal{M}_{\text{ren}}(M, m_t, t_1, \mu)\rangle \quad \text{[Neubert/Becher]}$$

$$\lim_{\epsilon\to 0} \mathbf{Z}^{-1}(\epsilon, M, t_1, \mu)\,|\mathcal{M}(\epsilon, M, t_1)\rangle = |\mathcal{M}_{\text{ren}}(M, t_1, \mu)\rangle \quad \text{[Catani; Neubert/Becher]}$$

Combine to get a relation between Z factors and finite matching function f

$$Z_{[q]}(\epsilon, m_t, \mu)\mathbf{Z}_m^{-1}(\epsilon, M, m_t, t_1, \mu) = f(m_t, \mu)\mathbf{Z}^{-1}(\epsilon, M, t_1, \mu)\,.$$

It follows that

$$\mathbf{H}_{ij}^m(M, m_t, t_1, \mu) = f^2(m_t, \mu)\mathbf{H}_{ij}(M, t_1, \mu) + \mathcal{O}\left(\frac{m_t^2}{\hat{s}}\right)\,.$$

Have factorized hard function in $m_t^2 \ll \hat{s}$ limit. Have checked this works to NLO, to NNLO for μ dependent terms, and that $f(m_t, \mu) = C_D(m_t, \mu)$.

Factorization for 1PI observables (i.e. $d\sigma/dp_T$)

1PI more involved than PIM case, just putting in fragmentation function doesn't work. We instead started from cross section in soft limit and derived small-mass limit from scratch

$$d\hat{\sigma}_{ij}(s_4, \hat{s}, \hat{t}_1, \hat{u}_1, m_t, \mu_f) = \mathrm{Tr}\left[\mathbf{H}^m_{ij}(\hat{s}, \hat{t}_1, \hat{u}_1, m_t, \mu_f)\, \mathbf{S}^{m,\mathrm{1PI}}_{ij}(s_4, \hat{s}, \hat{t}_1, \hat{u}_1, m_t, \mu_f)\right] + \; O\left(\frac{s_4}{m_t^2}\right)$$

- know from PIM case that $\mathbf{H}^m_{ij} = C_D^2(m_t, \mu)\mathbf{H}_{ij}$
- problem is now: how to factorize multiscale soft function into component parts?
- Dealt with in [arXiv:1310.3836]. We used method of regions to figure out structure then worked with operators to make more formal

Momentum regions and factorization

- Regions analysis shows that three momentum regions contribute to massive soft function in limit $m_t^2 \ll \hat{s}$

$$k_s^\mu \sim \frac{s_4}{\sqrt{\hat{s}}} \sim \sqrt{\hat{s}}\,\frac{s_4}{m_t^2}\,(\lambda^2,\lambda^2,\lambda^2) \qquad \text{(soft, wide angle)}$$

$$k_{sc}^\mu \sim \frac{s_4\,p_3^\mu}{\hat{s}} \sim \sqrt{\hat{s}}\,\frac{s_4}{m_t^2}\,(\lambda^2,\lambda^4,\lambda^3) \qquad \text{(soft, collinear to the top)}$$

$$k_{sc'}^\mu \sim \frac{s_4\,p_4^\mu}{m_t^2} \sim \sqrt{\hat{s}}\,\frac{s_4}{m_t^2}\,(\lambda^2,1,\lambda) \qquad \text{(soft, collinear to the anti-top)}$$

- This leads to factorized form (checked explicitly to NNLO)

$$\begin{aligned}\mathbf{S}_{ij}^m(s_4,\hat{s},\hat{t}_1,\hat{u}_1,m_t,\mu) = &\int d\omega_s\,d\omega_d\,d\omega_b\,\delta(s_4-\omega_s-\omega_d-\omega_b)\\ &\times \mathbf{S}_{ij}\left(\omega_s,\frac{\omega_s}{\sqrt{\hat{s}}},x_t,\mu\right) S_D\left(\omega_d,\frac{\omega_d\,m_t}{\hat{s}},\mu\right) S_B\left(\omega_b,\frac{\omega_b}{m_t},\mu\right)\\ &+\mathcal{O}(s_4/m_t^2)+\mathcal{O}(m_t^2/\hat{s})\end{aligned}$$

- Can see structure through regions calculation, but helps to have operator definitions in SCET to connect with literature (and do all orders 'proof')

Factorization of Wilson loop operator I

- operator definition of soft function:

$$\mathbf{S}^m(\omega, \hat{s}, \hat{t}_1, \hat{u}_1, m_t, \mu) = \frac{1}{d_R} \sum_X \langle 0|\mathbf{O}_s^{m\dagger}(0)|X\rangle \, \langle X|\mathbf{O}_s^m(0)|0\rangle \; \delta(\omega - 2p_4 \cdot p_X)$$

$$\mathbf{O}_s^m(x) = \left[\mathbf{S}_{v_1}^m \mathbf{S}_{v_2}^m \mathbf{S}_{v_3}^m \mathbf{S}_{v_4}^m\right](x); \quad \mathbf{S}_{v_i}^m(x) = \mathcal{P} \exp\left(ig_s \int_0^\infty ds \; v_i \cdot A^a(x + sv_i) \, \mathbf{T}_i^a \right)$$

- to factorize it, first decompose gluon field as

$$A^a \to A_s^a + A_{sc}^a + A_{sc'}^a$$

- then use Wilson line identity

$$\begin{aligned} &\mathcal{P}\exp\left[\int_a^b dx\,(A(x)+B(x))\right] \\ &\quad = \mathcal{P}\exp\left[\int_a^b dx\,A(x)\right]\;\mathcal{P}\exp\left[\int_a^b dx\left(\mathcal{P}e^{\int_a^x dx' A(x')}\right)^{-1} B(x)\left(\mathcal{P}e^{\int_a^x dx' A(x')}\right)\right] \end{aligned}$$

Factorization of Wilson loop operator II

- first define Wilson lines depending on a single mode

$$
\begin{aligned}
\mathbf{S}_{v_i}(x) &= \mathcal{P}\exp\left(ig_s\int_0^\infty ds\; v_i\cdot A_s^a(x+sv_i)\,\mathbf{T}_i^a\right),\\
\mathbf{Y}_{v_i}(x) &= \mathcal{P}\exp\left(ig_s\int_0^\infty ds\; v_i\cdot A_{sc}^a(x+sv_i)\,\mathbf{T}_i^a\right),\\
\mathbf{Y}'_{v_i}(x) &= \mathcal{P}\exp\left(ig_s\int_0^\infty ds\; v_i\cdot A_{sc'}^a(x+sv_i)\,\mathbf{T}_i^a\right)
\end{aligned}
$$

- can then show

$$
\mathbf{S}^m_{v_i}(x) = \mathcal{P}\exp\left(ig_s\int_0^\infty ds\; v_i\cdot[A_s^a + A_{sc}^a + A_{sc'}^a](x+sv_i)\;\mathbf{T}_i^a\right) = \mathbf{Y}_{v_i}(x)\,\tilde{\mathbf{S}}_{v_i}(x)\,\tilde{\mathbf{Y}}'_{v_i}(x)
$$

- after SCET field redefinitions [Bauer, Pirjol, Stewart '01] this turns into a simple product of non-interacting Wilson lines for each sector

The factorized matrix elements

- "massless soft function" (known to NNLO from [Ferroglia, BP, Yang '12])

$$\mathbf{S}\left(\omega_s, \frac{\omega_s}{\sqrt{\hat{s}}}, x_t, \mu\right) = \frac{1}{d_R} \sum_{X_s} \langle 0|\mathbf{O}_s^\dagger(0)|X_s\rangle \langle X_s|\mathbf{O}_s(0)|0\rangle \; \delta(\omega_s - \sqrt{\hat{s}}\, n_4 \cdot p_{X_s})$$

$$\mathbf{O}_s(x) = \left[\mathbf{S}_{n_1}\mathbf{S}_{n_2}\mathbf{S}_{n_3}\mathbf{S}_{n_4}\right](x)$$

- "soft fragmention function" (known to NNLO from [Melnikov, Mitov '04])

$$S_D\left(\omega_{sc}, \frac{\omega_{sc} m_t}{\hat{s}}, \mu\right) = \sum_{X_{sc}} \langle 0|O_{sc}^\dagger(0)|X_{sc}\rangle \langle X_{sc}|O_{sc}(0)|0\rangle \; \delta(\omega_{sc} - \sqrt{\hat{s}}\, \bar{n}_3 \cdot p_{X_{sc}})$$

$$O_{sc}(x) = Y_{v_3}^\dagger(x)\, Y_{\bar{n}_3}(x)$$

- "heavy quark jet function" (known to NNLO from [Jain, Stewart '08])

$$S_B\left(\omega_{sc'}, \frac{\omega_{sc'}}{m_t}, \mu\right) = \sum_{X_{sc'}} \langle 0|O_{sc'}^\dagger(0)|X_{sc'}\rangle \langle X_{sc'}|O_{sc'}(0)|0\rangle \; \delta(\omega_{sc'} - 2m_t v_4 \cdot p_{X_{sc'}})$$

$$O_{sc'}(x) = Y_{\bar{n}_4}'^\dagger(x)\, Y_{v_4}'(x)$$

THE FINAL RESULT FOR 1PI CROSS SECTIONS IN SOFT AND SMALL-MASS LIMIT

In Laplace space:

$$\begin{aligned}\tilde{c}_{ij}(N,\hat{s},\hat{t}_1,\hat{u}_1,m_t,\mu) = C_D^2\left(\ln\frac{m_t^2}{\mu^2},\mu\right)\mathrm{Tr}\left[H_{ij}\left(\ln\frac{\hat{s}}{\mu^2},x_t,\mu\right)\tilde{s}_{ij}\left(\ln\frac{\hat{s}}{\bar{N}^2\mu^2},x_t,\mu\right)\right] \\ \times\,\tilde{s}_D\left(\ln\frac{m_t}{\bar{N}\mu},\mu\right)\tilde{s}_B\left(\ln\frac{\hat{s}}{\bar{N}m_t\mu},\mu\right) \\ +\,\mathcal{O}\left(\frac{\hat{s}}{Nm_t^2}\right)+\mathcal{O}\left(\frac{m_t^2}{\hat{s}}\right)\end{aligned}$$

- cross section factorized into five one-scale functions, all known to NNLO
- can solve RG equations as usual to do double resummation of soft and small-mass logarithms

Summary and outlook

Summary:

- used SCET-based factorization formulas to derive double soft and small mass limit of differential cross sections
- components known to accuracy needed for NNLO virtual plus soft approximations plus NNLL resummation of soft and small-mass logs

To do:

- implement double small-mass and soft-gluon resummation numerically
- include electroweak corrections (important at high M)
- compare in detail with NLO parton shower programs and match to NNLO calculations once available
- deal with heavy-quark loops

backup slides

THE MASSLESS PIM SOFT FUNCTION

Basic object is Wilson loop

$$\mathbf{O}_s(x) = \left[\mathbf{S}_{n_1}\mathbf{S}_{n_2}\mathbf{S}_{n_3}\mathbf{S}_{n_4}\right](x)$$

built out of light-like Wilson lines

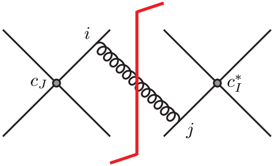
$$\mathbf{S}_i(x) = \mathcal{P}\exp\left(ig_s\int_{-\infty}^{0} ds\; n_i\cdot A^a(x+sn_i)\,\mathbf{T}_i^a\right)$$

Soft function is defined after squaring amplitude:

$$\mathbf{S}(\omega, t_1/M^2, \mu) = \frac{1}{d_R}\sum_{X_s}\langle 0|\mathbf{O}_s^\dagger(0)|X_s\rangle\,\langle X_s|\mathbf{O}_s(0)|0\rangle\;\delta(\omega-(n_1+n_2)\cdot p_{X_s})$$

- for two-to-two scattering $n_i\cdot n_j$ can be expressed in terms of one angle $-t_1/M^2$
- the δ-function sets $\omega = 2E_s = M(1-z)$ in partonic center-of-mass frame

Soft function at NLO



Feynman diagram proportional to (using dim. reg. in $d = 4 - 2\epsilon$ dimensions)

$$I_1(\omega, a_{ij}) = \int d^d k\, \delta(k^2)\, \theta(k^0)\, \frac{n_i \cdot n_j\; \delta(\omega - n_0 \cdot k)}{n_i \cdot k\;\; n_j \cdot k} \equiv \pi^{1-\epsilon}\, e^{-\epsilon\gamma_E}\, \omega^{-1-2\epsilon}\, \bar{I}_1(a_{ij});$$

$$a_{ij} \equiv 1 - \frac{n_0^2\;\, n_i \cdot n_j}{2\, n_0 \cdot n_i\;\, n_0 \cdot n_j}; \qquad \bar{I}_1(a) = \frac{2\, e^{\epsilon\gamma_E}\, \Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\, (1-a)^{-\epsilon}\; {}_2F_1(-\epsilon, -\epsilon, 1-\epsilon, a)$$

Bare soft function matrix obtained by summing over legs and evaluating color factors

$$\left[\mathbf{S}^{(1)}_{\text{bare}}\right]_{IJ} = \frac{2}{\omega} \left(\frac{\mu}{\omega}\right)^{2\epsilon} \sum_{\text{legs}} \langle c_I | \mathbf{T}_i \cdot \mathbf{T}_j | c_J \rangle\, \bar{I}_1(a_{ij})$$

Soft function at NNLO

Three basic types of topologies at NNLO

- two-Wilson-line graphs (depend on one scale a_{ij})
- three-Wilson-line graphs (depend on two scales a_{ij}, a_{ik})
- four-Wilson-line graphs (depend on two scales a_{ij}, a_{kl} but factorize)

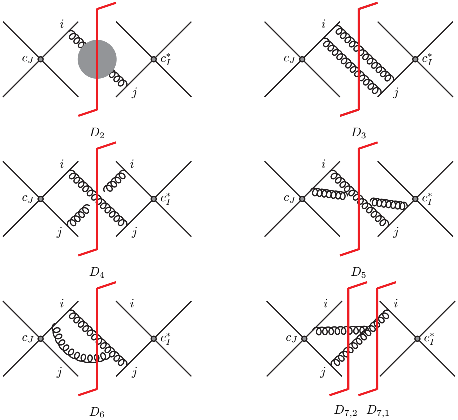
Especially the three-Wilson-line graphs are potentially very difficult, but turned out to be much simpler that we originally thought (fortuitous cancellations)!

Result at NNLO has form

$$\mathbf{S}^{(2)}_{\text{bare}} = \frac{4}{\omega}\left(\frac{\mu}{\omega}\right)^{4\epsilon}\sum_{\text{legs}}\left(\sum_{n=2}^{7}\mathbf{w}_{ij}^{(n)}\,\bar{I}_n(a_{ij}) + \mathbf{w}_{ijk}^{(8)}\,\bar{I}_8(a_{ij}, a_{ik}) + \mathbf{w}_{ijkl}^{(9)}\,\bar{I}_9(a_{ij}, a_{kl})\right)$$

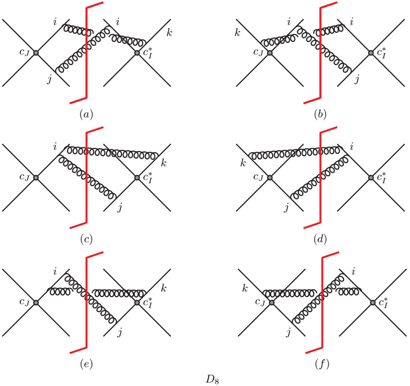
- integrals $\bar{I}_n$ solved in terms of various harmonic polylogarithms (HPLs)
- $1/\epsilon$ poles in bare function subtracted by renormalization procedure (derived from factorization so important cross check)

Two-wilson-line graphs



- parameterize phase-space in terms of light-cone coordinates
- hardest integrals evaluated by expanding in ϵ and deriving and solving differential equations w.r.t a_{ij} (checked with integrals in [Li, Mantry, Petriello '11])

Abelian three-Wilson-line graphs



Graphs (a) and (b) factorize.

Graphs (c)-(f) are complicated, but their sum factorizes

Final result for sum of graphs

$$D_8 \sim \{\mathbf{T}_i^a, \mathbf{T}_i^b\}\, \mathbf{T}_j^a\, \mathbf{T}_k^b \times (1 - a_{ij})^{-\epsilon}\, (1 - a_{ik})^{-\epsilon} \times {}_2F_1(-\epsilon, -\epsilon, 1 - \epsilon, a_{ij})\, {}_2F_1(-\epsilon, -\epsilon, 1 - \epsilon, a_{ik})$$

- factorized form follows from non-abelian exponentiation theorem, which states that coefficient of symmetric color structure is proportional to NLO graph squared

Cancellations

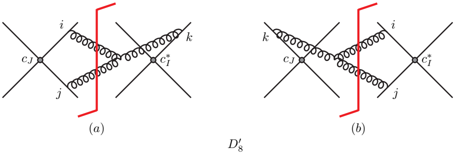


Figure : Examples of non-abelian three-Wilson-line integrals which add to zero.

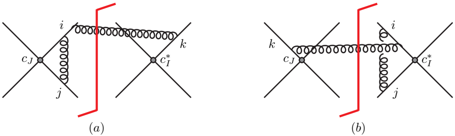


Figure : Example of a pair of mixed virtual-real one-particle cuts which adds up to a scaleless integral.

NLO momentum regions in small-mass limit

- Basic NLO integral for massive 1PI soft function

$$I_{ij}^m = \pi^{-1+\epsilon} e^{\epsilon\gamma_E} \mu^{2\epsilon} \int d^d k \delta^+(k^2)\, \delta^+(s_4 - 2p_4\cdot k)\, \frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}$$
$$\equiv \int [dk]\, \delta^+(s_4 - 2p_4\cdot k)\, \frac{p_i\cdot p_j}{p_i\cdot k\, p_j\cdot k}$$

- To identify momentum regions use light-cone coordinates:

$$k^\mu = k_{+ij}\, \frac{n_i^\mu}{\sqrt{2\, n_i\cdot n_j}} + k_{-ij}\, \frac{n_j^\mu}{\sqrt{2\, n_i\cdot n_j}} + k_{\perp ij}^\mu$$

- Using $i=3$ and $j=4$ have in partonic c.m. frame ($\lambda = m_t/\sqrt{\hat{s}}$) have scalings of $p_i^\mu = (p_{4+}, p_{4k-}, p_{4k\perp})$

$$p_3^\mu \sim \sqrt{\hat{s}}(1,\lambda^2,\lambda), \quad p_4^\mu \sim \sqrt{\hat{s}}(\lambda^2,1,\lambda); \quad p_3^2 = p_4^2 = m_t^2$$

$$p_1^\mu \sim \sqrt{\hat{s}}(1,1,1), \quad p_2^\mu \sim \sqrt{\hat{s}}(1,1,1); \quad p_1^2 = p_2^2 = 0$$

Regions I: Wide angle soft

Can approximate $p_i^\mu \sim E_i n_i^\mu$, $n_i^2 = 0$. The soft integral is then

$$
\begin{aligned}
I_{ij}^s &= \int [dk_s]\, \delta^+(s_4 - \sqrt{\hat{s}} n_4 \cdot k_s)\, \frac{n_i \cdot n_j}{n_i \cdot k_s\; n_j \cdot k_s} \\
&= \frac{1}{s_4} \left(\frac{s_4}{\sqrt{\hat{s}} \mu} \right)^{-2\epsilon} \left(\frac{2 n_i \cdot n_j}{n_4 \cdot n_i\; n_4 \cdot n_j} \right)^{-\epsilon} \left(-\frac{2}{\epsilon} + \frac{\pi^2}{6} \epsilon \right)
\end{aligned}
$$

- characteristic scale is $\mu_s \sim s_4/\sqrt{\hat{s}}$
- like an eikonal factor for "massless" partons
- total contribution from region after summing all over all legs:

$$
I^s = 2\mathbf{T}_1 \cdot \mathbf{T}_2\, I_{12}^s + 2\mathbf{T}_1 \cdot \mathbf{T}_3\, I_{13}^s + 2\mathbf{T}_2 \cdot \mathbf{T}_3\, I_{12}^s
$$

- matrix in color space, and emissions from parton 4 vanish

Regions II: soft and collinear to top quark

write $p_3^\mu = \sqrt{\hat{s}} n_3^\mu/2 + n_4^\mu m_t^2/2\sqrt{\hat{s}}$, so $v_3^2 = p_3^2/m_t^2 = 1$. Then

$$I^{sc}_{(i\neq 3)3} = \int [dk_{sc}]\, \delta^+(s_4 - \sqrt{\hat{s}} k^+_{sc}) \frac{2v_3^+}{(v_3^+ k^-_{sc} + v_3^- k^+_{sc})\, k^+_{sc}} = \frac{1}{s_4}\left(\frac{s_4 m_t}{\hat{s}\mu}\right)^{-2\epsilon}\left(\frac{1}{\epsilon} + \frac{\pi^2}{12}\epsilon\right)$$

$$I^{sc}_{33} = \int [dk_{sc}]\, \delta^+(s_4 - \sqrt{\hat{s}} k^+_{sc}) \frac{1}{v_3 \cdot k_{sc}\, v_3 \cdot k_{sc}} = \frac{2}{s_4}\left(\frac{s_4 m_t}{\hat{s}\mu}\right)^{-2\epsilon}$$

- characteristic scale is $\mu_{sc} \sim m_t s_4/\hat{s}$
- integrals for $i \neq 3$ do not depend on n_i
- total contribution from region

$$I^{sc} = \mathbf{T}_3 \cdot \mathbf{T}_3\, I^{sc}_{33} + 2 I^{sc}_{13} \sum_{i\neq 3} \mathbf{T}_i \cdot \mathbf{T}_3 = C_F (I^{sc}_{33} - 2 I^{sc}_{13})$$

- color diagonal! can show this is one loop contribution to soft part of fragmentation function, S_D

Regions III: Soft and collinear to antitop quark

Write $p_4^\mu = \sqrt{\hat{s}} n_4^\mu/2 + n_3^\mu m_t^2/2\sqrt{\hat{s}}$. Then

$$I^{sc'}_{(i\neq 4)4} = \int [dk_{sc'}]\, \delta^+(s_4 - 2m_t v_4 \cdot k_{sc'})\, \frac{v_4^-}{v_4 \cdot k_{sc'}\;\, k^-_{sc'}} = \frac{1}{s_4}\left(\frac{s_4}{m_t \mu}\right)^{-2\epsilon}\left(-\frac{1}{\epsilon} + \frac{\pi^2}{4}\epsilon\right)$$

$$I^{sc'}_{44} = \int [dk_{sc'}]\, \delta^+(s_4 - 2m_t v_4 \cdot k_{sc'})\, \frac{1}{v_4 \cdot k_{sc'}\;\, v_4 \cdot k_{sc'}} = \frac{1}{s_4}\left(\frac{s_4}{m_t \mu}\right)^{-2\epsilon}(2 + 4\epsilon)$$

- characteristic scale is $\mu_{sc'} \sim s_4/m_t$
- integrals for $i \neq 4$ do not depend on n_i
- total contribution from region

$$I^{sc'} = \mathbf{T}_4 \cdot \mathbf{T}_4\, I^{sc'}_{44} + 2 I^{sc'}_{14} \sum_{i\neq 4} \mathbf{T}_i \cdot \mathbf{T}_4 = C_F (I^{sc'}_{44} - 2 I^{sc'}_{14})$$

- color diagonal! can show this is one loop contribution to the heavy-quark jet function, S_B [Fleming, Mantry, Hoang, Stewart '08]